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Relativistic Landau problem at finite temperature*

C G Beneventano[1] **and E M Santangelo**[2]

Departamento de Física, Universidad Nacional de La Plata, Instituto de Física de La Plata, UNLP-CONICET, CC 67, 1900 La Plata, Argentina

E-mail: gabriela@obelix.fisica.unlp.edu.ar and mariel@obelix.fisica.unlp.edu.arw

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Abstract
We study the zero temperature Casimir energy and fermion number for Dirac fields in a (2+1)-dimensional Minkowski space-time, in the presence of a uniform magnetic field perpendicular to the spatial manifold. Then, we go to the finite-temperature problem, with a chemical potential, introduced as a uniform zero component of the gauge potential. By performing a Lorentz boost, we obtain the Hall conductivity in the case of crossed electric and magnetic fields.

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1. Introduction

The quantization of the Hall conductivity [1] is a remarkable quantum phenomenon, which occurs in two-dimensional electron systems, at low temperatures and strong perpendicular magnetic fields. Most proposed explanations for this phenomenon [2] rely on Schröedinger's one-particle theory and the introduction of a filling fraction of the Landau levels, which must be assumed to be integer to reproduce the observed behaviour.

It is the aim of this paper to show that, in the context of relativistic field theory, such behaviour arises naturally, as a consequence of the spin-statistics theorem.

In section 2 we present the theory of Dirac fields in a (2 + 1) Minkowski space-time, interacting with a magnetic background field perpendicular to the spatial plane, and evaluate the vacuum expectation values of the energy and fermion density. Section 3 contains the generalities of the same theory in an Euclidean three-dimensional space, and presents the eigenvalues of the corresponding Dirac operator. From such eigenvalues, the partition function is evaluated in section 4. Section 5 contains the resulting free energy and mean particle density

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[1] Member of CONICET.
[2] Member of CONICET.

at finite temperature. Finally, in section 6 we perform an adequate Lorentz boost in order to consider the problem of fermion interaction with crossed electric and magnetic fields, and obtain the Hall conductivity.

2. Zero-temperature problem

We study a (2 + 1)-dimensional theory of Dirac fields, in the presence of a uniform background magnetic field perpendicular to the spatial plane. We choose the metric $(-, +, +)$, natural units $\hbar = c = 1$, and adopt the following representation for the Dirac matrices: $\gamma_M^0 = \mathrm{i}\sigma_3$, $\gamma_M^1 = \sigma_2$ and $\gamma_M^2 = \sigma_1$.

The Hamiltonian can be determined from the solutions of the Dirac equation $(\mathrm{i}\not{\partial} - e\not{A})\Psi = 0$, where $-e$ is the negative charge of the electron. In the Landau gauge $A = (0, 0, Bx)$, with $B > 0$. Thus, after setting $\Psi(t, x, y) = \mathrm{e}^{-\mathrm{i}Et}\psi(x, y)$, we get the Hamiltonian $H = \mathrm{i}\sigma_1\partial_x - \mathrm{i}\sigma_2\partial_y + \sigma_2 eBx$.

In order to solve the eigenvalue problem for this Hamiltonian, we take

$$\psi_k(x, y) = \begin{pmatrix} \varphi_k(x, y) \\ \chi_k(x, y) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}ky}\varphi_k(x) \\ \mathrm{e}^{\mathrm{i}ky}\chi_k(x) \end{pmatrix}. \tag{1}$$

This leads to the following system of first order equations:

$$(\mathrm{i}\partial_x - \mathrm{i}k - \mathrm{i}eBx)\chi_k = E\varphi_k \qquad (\mathrm{i}\partial_x + \mathrm{i}k + \mathrm{i}eBx)\varphi_k = E\chi_k. \tag{2}$$

After imposing that the eigenfunctions be well behaved for $\rightarrow \pm\infty$, we find two types of solutions to our problem[3]:

1. A definite-chirality zero mode ($E_0 = 0$).
2. A set of eigenfunctions corresponding to the symmetric spectrum $E_n = \pm\sqrt{2neB}$, $n = 1, \ldots, \infty$.

In all cases, the eigenfunctions can be written in terms of Hermite polynomials, and all eigenvalues exhibit the well-known Landau degeneracy per unit area:

$$\Delta_L = \frac{eB}{2\pi}. \tag{3}$$

The vacuum expectation value of the energy per unit area, defined through a zeta-function regularization (see, for example, [3] and references therein), is given by $E_C = -\frac{\Delta_L}{2}\sum_{E_n\neq 0}|E_n|^{-s}\big\rfloor_{s=-1}$.

In the present case, we have (α is an arbitrary parameter with mass dimension, introduced to render the complex powers dimensionless)

$$E_C(B) = -\frac{\Delta_L\alpha}{2} 2\sum_{n=1}^{\infty}\left(\frac{\sqrt{2neB}}{\alpha}\right)^{-s}\Bigg\rfloor_{s=-1} = -\Delta_L\sqrt{2eB}\,\zeta_R\left(-\frac{1}{2}\right). \tag{4}$$

Always in the zeta-function regularization framework, the fermion number is [4]

$$N(B) = -\frac{\Delta_L}{2}\left(\sum_{E_n>0}|E_n|^{-s} - \sum_{E_n<0}|E_n|^{-s}\right)\Bigg\rfloor_{s=0} + N_0,$$

where N_0 is the contribution coming from zero modes.

[3] For the other non-equivalent representation of the gamma matrices in 2 + 1 dimensions, the spectrum does not change.

In our case, the nonvanishing spectrum is symmetric. So, only the zero mode, which is charge self-conjugate, contributes. This gives as a result [4]

$$N(B) = \pm\frac{\Delta_L}{2}. \tag{5}$$

Or, equivalently, for the vacuum expectation value of the charge density

$$j^0(B) = \mp e\frac{\Delta_L}{2}. \tag{6}$$

3. The theory at finite temperature with chemical potential

In order to study the effect of temperature, we go to an Euclidean space, with the metric (+, +, +). To this end, we take the Euclidean gamma matrices to be $\gamma_0 = \mathrm{i}\gamma_M^0 = -\sigma_3$, $\gamma_1 = \gamma_M^1 = \sigma_2$, $\gamma_2 = \gamma_M^2 = \sigma_1$. We will follow [5] in introducing the chemical potential as an imaginary $A_0 = -\mathrm{i}\frac{\mu}{e}$ in the Euclidean space. Thus, the partition function in the grand-canonical ensemble is given by

$$\ln Z = \ln\det(\mathrm{i}\partial\!\!\!/ - eA\!\!\!/). \tag{7}$$

In order to evaluate it in the zeta regularization approach [6], we first determine the eigenfunctions, and the corresponding eigenvalues, of the Dirac operator, in the same gauge used in the previous section, i.e., we solve

$$[-\mathrm{i}\sigma_3(\partial_\tau + \mu) + \mathrm{i}\sigma_2\partial_x + \sigma_1(\mathrm{i}\partial_y - eBx)]\Psi = \omega\Psi. \tag{8}$$

To satisfy antiperiodic boundary conditions in the τ direction, we propose

$$\Psi_{k,l}(\tau, x, y) = \frac{\mathrm{e}^{\mathrm{i}\lambda_l\tau}\,\mathrm{e}^{\mathrm{i}ky}}{\sqrt{2\pi\beta}}\psi_{k,l}(x), \qquad \text{with} \quad \lambda_l = (2l+1)\frac{\pi}{\beta}, \tag{9}$$

where $\beta = \frac{1}{T}$ is the inverse temperature.

After doing so, and writing $\psi_{k,l}(x) = \binom{\varphi_{k,l}(x)}{\chi_{k,l}(x)}$, we have for each k, l,

$$\begin{aligned}(\partial_x - k - eBx)\chi_{k,l} &= (\omega - \tilde{\lambda}_l)\varphi_{k,l}\\ (-\partial_x - k - eBx)\varphi_{k,l} &= (\omega + \tilde{\lambda}_l)\chi_{k,l},\end{aligned} \tag{10}$$

where $\tilde{\lambda}_l = \lambda_l - \mathrm{i}\mu$.

There are two types of eigenvalues

1. $\omega_l = \tilde{\lambda}_l$, with $l = -\infty, \ldots, \infty$.
2. $\omega_{l,n} = \pm\sqrt{\tilde{\lambda}_l^2 + 2neB}$, with $n = 1, \ldots, \infty$, $l = -\infty, \ldots, \infty$.

In all cases, the degeneracy per unit area is again given by Δ_L in equation (3).

4. Evaluation of the partition function at finite temperature and chemical potential

The partition function, in the zeta regularization scheme [3], is given by

$$\log \mathcal{Z} = -\frac{\mathrm{d}}{\mathrm{d}s}\bigg\rfloor_{s=0} \zeta\left(s, \frac{\mathrm{i}\partial\!\!\!/ - eA\!\!\!/}{\alpha}\right). \tag{11}$$

As in the previous section, α is a parameter with mass dimension, introduced to render the ζ-function dimensionless.

We must consider two contributions to $\log \mathcal{Z}$, respectively coming from eigenvalues of type 1 and 2 in the previous section, i.e.,

$$\begin{aligned}\Delta_1(\mu) &= -\frac{\mathrm{d}}{\mathrm{d}s}\bigg\rfloor_{s=0} \zeta_1(s,\mu) \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\bigg\rfloor_{s=0} \Delta_L \sum_{l=-\infty}^{\infty}\left[(2l+1)\frac{\pi}{\alpha\beta} - \mathrm{i}\frac{\mu}{\alpha}\right]^{-s},\end{aligned} \tag{12}$$

and

$$\begin{aligned}\Delta_2(\mu,B) &= -\frac{\mathrm{d}}{\mathrm{d}s}\bigg\rfloor_{s=0} \zeta_2(s,\mu,B) \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\bigg\rfloor_{s=0} (1+(-1)^{-s})\Delta_L \sum_{\substack{n=1\\ l=-\infty}}^{\infty}\left[\frac{2neB}{\alpha^2} + \left((2l+1)\frac{\pi}{\alpha\beta} - \mathrm{i}\frac{\mu}{\alpha}\right)^2\right]^{-\frac{s}{2}}.\end{aligned} \tag{13}$$

In the rest of this section, we sketch the main steps in the analytic extension of both zeta functions and in the calculation of their s-derivatives (for a detailed presentation, see [7]). The contribution $\Delta_1(\mu)$ can be evaluated at once for the whole μ-range. The analytic extension of $\zeta_1(s,\mu)$ can be achieved as follows (for a similar calculation, see [8]):

$$\zeta_1(s,\mu) = \Delta_L\left(\frac{2\pi}{\alpha\beta}\right)^{-s}\left[\zeta_H\left(s,\frac{1}{2}-\frac{\mathrm{i}\mu\beta}{2\pi}\right) + \sum_{l=0}^{\infty}\left[-\left(l+\frac{1}{2}\right) - \mathrm{i}\frac{\mu\beta}{2\pi}\right]^{-s}\right]. \tag{14}$$

Now, in order to write the second term as a Hurwitz zeta, we must relate the eigenvalues with negative real part to those with positive one without, in so doing, going through zeros in the argument of the power. Otherwise stated, we must select a cut in the complex ω plane [9]. This requirement determines a definite value of $(-1)^{-s}$, i.e., $(-1)^{-s} = \mathrm{e}^{\mathrm{i}\pi\,\mathrm{sign}(\mu)s}$. Taking this into account, we finally have

$$\zeta_1(s,\mu) = \Delta_L\left(\frac{2\pi}{\beta\alpha}\right)^{-s}\left[\zeta_H\left(s,\frac{1}{2}-\frac{\mathrm{i}\mu\beta}{2\pi}\right) + \mathrm{e}^{\mathrm{i}\pi\,\mathrm{sign}(\mu)s}\zeta_H\left(s,\frac{1}{2}+\frac{\mathrm{i}\mu\beta}{2\pi}\right)\right]. \tag{15}$$

From this last expression, the contribution $\Delta_1(\mu)$ to $\log \mathcal{Z}$ can be obtained. It is given by

$$\begin{aligned}\Delta_1(\mu) &= -\Delta_L\left[\zeta_H'\left(0,\frac{1}{2}-\frac{\mathrm{i}\mu\beta}{2\pi}\right) + \zeta_H'\left(0,\frac{1}{2}+\frac{\mathrm{i}\mu\beta}{2\pi}\right) + \mathrm{i}\pi\,\mathrm{sign}(\mu)\zeta_H\left(0,\frac{1}{2}+\frac{\mathrm{i}\mu\beta}{2\pi}\right)\right] \\ &= \Delta_L\left\{\log\left(2\cosh\left(\frac{\mu\beta}{2}\right)\right) - \frac{|\mu|\beta}{2}\right\}.\end{aligned} \tag{16}$$

The analytic extension of $\zeta_2(s,\mu,B)$ requires a separate consideration of different μ ranges. We study in detail two of these ranges.

4.1. $\mu^2 < 2eB$

$$\zeta_2(s,\mu,B) = (1+(-1)^{-s})\Delta_L \sum_{\substack{n=1\\ l=-\infty}}^{\infty}\left[\frac{2neB}{\alpha^2} + \left((2l+1)\frac{\pi}{\alpha\beta} - \mathrm{i}\frac{\mu}{\alpha}\right)^2\right]^{-\frac{s}{2}}. \tag{17}$$

Making use of the Mellin transform, this can be written as

$$\begin{aligned}\zeta_2(s,\mu,B) = \frac{(1+(-1)^{-s})\Delta_L}{\Gamma\left(\frac{s}{2}\right)} \sum_{n=1}^{\infty}\int_0^{\infty}\mathrm{d}t\,t^{\frac{s}{2}-1}\exp\left(-t\left[\frac{2neB}{\alpha^2} + \left(\frac{\pi}{\alpha\beta} - \frac{\mathrm{i}\mu}{\alpha}\right)^2\right]\right) \\ \times\Theta_3\left(\frac{-2t}{\alpha\beta}\left(\frac{\pi}{\alpha\beta} - \frac{\mathrm{i}\mu}{\alpha}\right), \frac{4\pi t}{(\alpha\beta)^2}\right),\end{aligned} \tag{18}$$

where we have used the definition of the Jacobi theta function $\Theta_3(z, x) = \sum_{l=-\infty}^{\infty} \mathrm{e}^{-\pi x l^2} \times \mathrm{e}^{2\pi z l}$.

To proceed, we use the inversion formula for the Jacobi function,
$\Theta_3(z, x) = \frac{1}{\sqrt{x}} \exp\left(\frac{\pi z^2}{x}\right) \Theta_3\left(\frac{z}{\mathrm{i}x}, \frac{1}{x}\right)$, and perform the integration over t, thus getting

$$\zeta_2(s, \mu, B) = \frac{(1+(-1)^{-s})\Delta_L \alpha\beta s}{4\sqrt{\pi}\Gamma\left(\frac{s+2}{2}\right)} \left[\Gamma\left(\frac{s-1}{2}\right) \left(\frac{2eB}{\alpha^2}\right)^{\frac{1-s}{2}} \zeta_R\left(\frac{s-1}{2}\right) \right.$$
$$\left. + 4 \sum_{n,l=1}^{\infty} (-1)^l . \left(\frac{l^2\alpha^4\beta^2}{8neB}\right)^{\frac{s-1}{4}} \cosh(\mu\beta l) K_{\frac{s-1}{2}}(\sqrt{2neB\beta^2 l^2}) \right]. \tag{19}$$

From this expression, the contribution Δ_2 to the partition function can be readily obtained, since the factor accompanying s is finite at $s = 0$. After using that $K_{-\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\, \mathrm{e}^{-x}$, and performing the resulting sum over l, we obtain

$$\Delta_2(\mu, B) = \Delta_L \beta \left[\sqrt{2eB}\zeta_R\left(-\frac{1}{2}\right) + \frac{1}{\beta} \sum_{n=1}^{\infty} \log\left(1 + \mathrm{e}^{-2\sqrt{2neB}\beta} + 2\cosh(\mu\beta)\mathrm{e}^{-\sqrt{2neB}\beta}\right) \right]. \tag{20}$$

Finally, adding the contributions given by equations (16) and (20) we get, for the partition function in the range $\mu^2 \leqslant 2eB$,

$$\log Z = \Delta_L \left\{ \log\left(2\cosh\left(\frac{\mu\beta}{2}\right)\right) - \frac{|\mu|\beta}{2} + \beta\sqrt{2eB}\zeta_R\left(-\frac{1}{2}\right) \right.$$
$$\left. + \sum_{n=1}^{\infty} \log\left(1 + \mathrm{e}^{-2\sqrt{2neB}\beta} + 2\cosh(\mu\beta)\,\mathrm{e}^{-\sqrt{2neB}\beta}\right) \right\}. \tag{21}$$

4.2. $2eB < \mu^2 < 4eB$

As before, we have

$$\zeta_2(s, \mu, B) = (1+(-1)^{-s})\Delta_L \sum_{\substack{n=1\\ l=-\infty}}^{\infty} \left[\frac{2neB}{\alpha^2} + \left((2l+1)\frac{\pi}{\alpha\beta} - \mathrm{i}\frac{\mu}{\alpha}\right)^2 \right]^{-\frac{s}{2}}. \tag{22}$$

However, in this range of μ, the contribution to the zeta function due to $n = 1$ must be analytically extended in a different way. In fact, the expression cannot be written in terms of a unique Mellin transform, since its real part is not always positive (note, in connection with this, for $n = 1$, equation (19) diverges). Instead, it can be written as a product of two Mellin transforms

$$\zeta_2^{n=1}(s, \mu, B) = \frac{(1+(-1)^{-s})}{\alpha^{-s}\left[\Gamma\left(\frac{s}{2}\right)\right]^2} \Delta_L \sum_{l=0}^{\infty} \int_0^{\infty} \mathrm{d}t\, t^{\frac{s}{2}-1} \exp\left(-\left[(2l+1)\frac{\pi}{\beta} - \mathrm{i}\mu + \mathrm{i}\sqrt{2eB}\right] t\right)$$
$$\times \int_0^{\infty} \mathrm{d}z\, z^{\frac{s}{2}-1} \exp\left(-\left[(2l+1)\frac{\pi}{\beta} - \mathrm{i}\mu - \mathrm{i}\sqrt{2eB}\right] z\right) + \mu \to -\mu \tag{23}$$

or, after changing variables according to $t' = t - z$; $z' = t + z$, performing one of the integrals, and the sum over l

$$\zeta_2^{n=1}(s, \mu, B) = \frac{(1+(-1)^{-s})\sqrt{\pi}}{2\alpha^{-s}\Gamma\left(\frac{s}{2}\right)} \Delta_L \left(2\sqrt{2eB}\right)^{\frac{1-s}{2}}$$
$$\times \int_0^{\infty} \mathrm{d}z\, z^{\frac{s-1}{2}} J_{\frac{s-1}{2}}(\sqrt{2eB}z) \frac{\mathrm{e}^{\mathrm{i}\mu z}}{\sinh\left(\frac{\pi z}{\beta}\right)} + \mu \to -\mu. \tag{24}$$

Now, the integral in this expression diverges at $z = 0$. In order to isolate such divergence, we add and subtract the first term in the series expansion of the Bessel function, thus getting the following two pieces:

$$\zeta_{2,(1)}^{n=1}(s,\mu,B) = \frac{(1+(-1)^{-s})\sqrt{\pi}s}{4\alpha^{-s}\Gamma\left(\frac{s}{2}+1\right)}\Delta_L(2\sqrt{2eB})^{\frac{1-s}{2}} \times \int_0^\infty \mathrm{d}z\, z^{\frac{s-1}{2}}\left[J_{\frac{s-1}{2}}(\sqrt{2eB}z) - \frac{\left(\frac{\sqrt{2eB}z}{2}\right)^{\frac{s-1}{2}}}{\Gamma\left(\frac{s+1}{2}\right)}\right]\frac{\mathrm{e}^{\mathrm{i}\mu z}}{\sinh\left(\frac{\pi z}{\beta}\right)} + \mu \to -\mu, \tag{25}$$

and

$$\zeta_{2,(2)}^{n=1}(s,\mu,B) = \frac{(1+(-1)^{-s})\sqrt{\pi}}{2^s\alpha^{-s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}\Delta_L\int_0^\infty \mathrm{d}z\, z^{s-1}\frac{\mathrm{e}^{\mathrm{i}\mu z}}{\sinh\left(\frac{\pi z}{\beta}\right)} + \mu \to -\mu. \tag{26}$$

The contribution of equation (25) to the partition function can be easily evaluated by noting that the factor multiplying s is finite at $s = 0$. This gives

$$\Delta_{2,(1)}^{n=1}(\mu,B) = \Delta_L\left\{\log\left(1+\mathrm{e}^{-2|\mu|\beta} + 2\cosh(\sqrt{2eB}\beta)\,\mathrm{e}^{-|\mu|\beta}\right) + |\mu|\beta - 2\log\left(2\cosh\left(\frac{\mu\beta}{2}\right)\right)\right\}. \tag{27}$$

In order to get the contribution coming from (26), the integral can be evaluated for $\Re s > 1$, which gives

$$\zeta_{2,(2)}^{n=1}(s,\mu,B) = \frac{(1+(-1)^{-s})\Gamma(s)\sqrt{\pi}(\alpha\beta)^s\Delta_L}{(2\pi)^s2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)} \times\left[\zeta_H\left(s,\frac{1}{2}\left(1-\frac{\mathrm{i}\mu\beta}{\pi}\right)\right) + \zeta_H\left(s,\frac{1}{2}\left(1+\frac{\mathrm{i}\mu\beta}{\pi}\right)\right)\right], \tag{28}$$

where $\zeta_H(s,x)$ is the Hurwitz zeta function. Its contribution to the partition function can now be evaluated by using that $\zeta_H\left(0,\frac{1}{2}\left(1-\frac{\mathrm{i}\mu\beta}{\pi}\right)\right)+\zeta_H\left(0,\frac{1}{2}\left(1+\frac{\mathrm{i}\mu\beta}{\pi}\right)\right) = 0$ and the well-known value of $-\frac{\mathrm{d}}{\mathrm{d}s}\big\rfloor_{s=0}\zeta_H(s,x)$ [10], to obtain

$$\Delta_{2,(2)}^{n=1}(\mu,B) = 2\Delta_L\log\left(2\cosh\left(\frac{\mu\beta}{2}\right)\right). \tag{29}$$

Summing up the contributions in equations (16), (27) and (29), as well as the contribution coming from $n \geqslant 2$, evaluated as in the previous subsection, one gets for the partition function

$$\log Z = \Delta_L\left\{\log\left(2\cosh\left(\frac{\mu\beta}{2}\right)\right) + \frac{|\mu|\beta}{2} + \log\left(1+\mathrm{e}^{-2|\mu|\beta}+2\cosh(\sqrt{2eB}\beta)\,\mathrm{e}^{-|\mu|\beta}\right) + \beta\sqrt{2eB}\left(\zeta_R\left(-\frac{1}{2}\right)-1\right) + \sum_{n=2}^{\infty}\log\left(1+\mathrm{e}^{-2\sqrt{2neB}\beta}+2\cosh(\mu\beta)\,\mathrm{e}^{-\sqrt{2neB}\beta}\right)\right\}. \tag{30}$$

At first sight, this result looks different from the one corresponding to $\mu^2 < 2eB$ (equation (21)). However, it is easy to see that both expressions coincide. The advantage of using expression (30) for this range of μ is that the zero-temperature limit is explicitly isolated from finite-temperature corrections.

5. Free energy and particle number

From equations (21) and (30), the free energy per unit area $\left(F = -\frac{1}{\beta}\log Z\right)$ can be obtained[4]. It is given by

$$F(\mu, B, \beta) = -\Delta_L\left\{\frac{1}{\beta}\log\left(2\cosh\left(\frac{\mu\beta}{2}\right)\right) - \frac{|\mu|}{2} + \sqrt{2eB}\zeta_R\left(-\frac{1}{2}\right)\right.$$
$$\left. + \frac{1}{\beta}\sum_{n=1}^{\infty}\log\left(1 + \mathrm{e}^{-2\sqrt{2neB}\beta} + 2\cosh(\mu\beta)\,\mathrm{e}^{-\sqrt{2neB}\beta}\right)\right\}. \qquad (31)$$

Moreover, the free energy is continuous at $\mu^2 = 2neB, n = 0, \ldots, \infty$. In the low-temperature limit one has

$$F(\mu^2 < 2eB) \rightarrow_{\beta\rightarrow\infty} -\Delta_L\sqrt{2eB}\zeta_R\left(-\frac{1}{2}\right),$$

which coincides with the Casimir energy obtained in section 2, even for $\mu \neq 0$, but in this range, i.e., for μ less than the first Landau level, if positive, or greater than minus the first Landau level, if negative. On the other hand,

$$F(2eB < \mu^2 < 4eB) \rightarrow_{\beta\rightarrow\infty} -\Delta_L\left\{\sqrt{2eB}\left(\zeta_R\left(-\frac{1}{2}\right) - 1\right) + |\mu|\right\}.$$

The mean particle density can be obtained as $N = \frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}i\mu}\log Z$. For nonzero temperature and arbitrary μ (not coinciding with an energy level[5]) one has

$$N(\mu, B, \beta) = \Delta_L\left\{\frac{1}{2}\left[\tanh\left(\frac{\mu\beta}{2}\right) - \mathrm{sign}(\mu)\right]\right.$$
$$\left. + \sum_{n=1}^{\infty}\frac{2\sinh(\mu\beta)\,\mathrm{e}^{-\sqrt{2neB}\beta}}{1 + \mathrm{e}^{-2\sqrt{2neB}\beta} + 2\cosh(\mu\beta)\,\mathrm{e}^{-\sqrt{2neB}\beta}}\right\}. \qquad (32)$$

It is interesting to note that, for $\mu = 0$, one has $N(\mu = 0) = \pm\frac{\Delta_L}{2}$, which shows that, in the absence of chemical potential, the fermion number obtained in equation (5) remains unaltered as the temperature grows.

On the other hand, for nonvanishing μ, the low-temperature limit differs, depending on the μ-range considered

$$N(2eBn < \mu^2 < 2eB(n+1)) \rightarrow_{\beta\rightarrow\infty} n\Delta_L\mathrm{sign}(\mu),$$

where $n = \left[\frac{\mu^2}{2eB}\right]$.

This result is nothing but the expected one for particles with the statistics of fermions, since relativistic field theory naturally leads to the spin-statistics theorem. At zero temperature, μ is nothing but the Fermi energy; for example, for $\mu > 0$, as μ grows past a Landau level, such a level becomes entirely filled.

[4] Consistently with the footnote in section 2, all the results in this section are independent of the representation of the gamma matrices chosen.

[5] Note that, for instance, if $\mu = 2neB$, the series in equation (19) converges only conditionally, and its term-by-term derivative leads to a divergent series.

6. Final comments

From the previous result, the mean value of the particle density at zero temperature can be obtained. After recovering units, one has

$$j^0(2ec^2\hbar Bn < \mu^2 < 2eBc^2\hbar(n+1)) = \frac{-nce^2B}{h}\mathrm{sign}(\mu),$$

the other two components of the current density tri-vector being equal to zero in the absence of an electric field.

Now, the zero-temperature limit of the same tri-vector in the presence of crossed homogeneous electric (F') and magnetic (B') fields can be retrieved, for $F' < cB'$, by performing a Lorentz boost with absolute value of the velocity $\frac{F'}{B'}$. Suppose, for definiteness, that the homogeneous electric field points along the positive y axis. Then, the velocity of the Lorentz boost must point along the negative x-axis, and the transformation gives as a result

$$j'^0 = \frac{-nce^2B'}{h}\mathrm{sign}(\mu), \qquad j'^x = \frac{-ne^2F'}{h}\mathrm{sign}(\mu), \qquad j'^y = 0.$$

As a consequence, the quantized zero-temperature Hall conductivity is

$$\sigma_{xy} = \frac{-ne^2}{h}\mathrm{sign}(\mu).$$

Finally, we mention that the more realistic case of massive fermions is at present under study [11].

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